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1997 J. Phys. A: Math. Gen. 30 6233

(http://iopscience.iop.org/0305-4470/30/18/009)

J. Phys. A: Math. Gen. **30** (1997) 6233–6244. Printed in the UK PII: S0305-4470(97)80475-1

Phase transitions in Scheidegger and Eden networks

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Received 16 December 1996, in final form 11 April 1997

Abstract. We study multifractal scaling and phase transitions in Scheidegger and Eden networks in the plane on several lattices. The Horton constants R_B and R_L are found not to depend on the lattice. The scaling exponent α in the integrated area distribution function $P(A > a) \sim a^{-\alpha}$ is found to be consistent with the relation $\alpha = 1 - \log R_L / \log R_B$. The exponent α defines a phase transition point in the multifractal spectrum of the area distribution. The approach to this phase transition is slow and controlled by $\ln M$, where M is the number of lattice points. For the Scheidegger model we are able to calculate the exact probability distribution $p(a)$ for small areas a and thus to study the finite-size scaling.

1. Introduction

Randomly branched structures appear in nature in a variety of circumstances, including river networks (Abrahams 1984, Tarboton *et al* 1988), the vascular tree of plants and animals (Suwa and Takahashi 1963, Zamir and Phillips 1988) and the branched patterns in deposition processes (Matsushita *et al* 1985). Ever since the discovery of fractal patterns in the model of diffusion limited aggregation (DLA) of Witten and Sander (1981) many models have been proposed and studied (Meakin *et al* 1991, Inaoka 1993, Kramer and Marder 1992). Most characterizations of the DLA-clusters have focused on the multifractal properties of the growth probability distribution (Turkevich and Scher 1985). More recently there has also been interest in the topology of the cluster, which has been characterized in terms of Horton's laws of stream numbers and stream length (Yekutieli *et al* 1994, Hinrichsen *et al* 1989).

In draining networks it was observed that the distribution of discharge scales algebraically and that the exponent is related to the Horton constants (de Vries *et al* 1994). This algebraic scaling is reminiscent of a non-hyperbolic phase in the appropriate measure and thus should give rise to a phase transition (Bohr *et al* 1988, Procaccia and Zeitak 1988). Previous work by Nagatani (1993a, b) on Scheidegger models has been inconclusive. We here study two models where we can identify the phase transition, study the finite-size scaling and verify a relation between the algebraic scaling and the Horton constants. The models we use generate compact and self-affine networks; they are the Scheidegger river network model (Scheidegger 1967) and a kind of Eden model first proposed by Meakin (1992).

This paper is organized as follows. In section 2 we introduce the models and methods to characterize their structure. We will give the analytical calculation of the multifractal spectrum for the area distribution and study its finite-size scaling behaviour. In section 3 we present numerical results on the two models on various lattices. This is followed by a

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calculation of the exact distribution function for the areas of the networks in the Scheidegger model and an analysis of its asymptotic algebraic decay in section 4. We close this paper with a summary of the main results and a discussion of open questions.

2. Models and methods

2.1. The Scheidegger and Eden networks

The computer simulation of the Scheidegger model is executed with the following simple algorithm. Consider a triangular lattice on a semi-infinite cylinder, stretched out in the plane with periodic lateral boundary conditions. The points on the bottom of the cylinder are defined as sinks. The network grows layer by layer by connecting points on the next layer randomly to points in the layer below. This way one obtains a directed network where every lattice point is connected in a unique way to the bottom of the lattice.

The second model is based on the Eden model (Eden 1960). The simulation starts from a seed particle in the middle or a seed line at the bottom of the lattice. The cluster grows by adding of randomly chosen perimeter sites to the cluster. The perimeter is defined as the set of unoccupied next-neighbour points. After an unoccupied perimeter site has been added, the direction connecting that site to one of its occupied neighbours is selected at random. All probabilities involved are uniform and one over the number of unoccupied sites or neighbours on the network. This way a tree is constructed that connects every site of the lattice by a unique path with the initial seed site or seed line.

For both models we used three different underlying lattices. The triangular lattice (lattice A), the square lattice (lattice B) and the square lattice with diagonals (lattice C) (figure 1). For the simulation of the Scheidegger model on lattices B and C one has to construct the network row by row from the bottom to the top by randomly choosing all sites of a row avoiding traps or crossings of paths. Paths which cross also have to be avoided during the simulation of the Eden model on lattices A and C.

In both models we are able to assign a weight $a(i)$ to every lattice point i which counts the number of sites which are connected to the bottom via this site. This weight equals the area of the corresponding subnetwork which is connected with the bottom through this site.

Examples of the resulting networks on a 100×100 square lattice are shown in figures 2 and 3. The thickness of the lines is proportional to the weight of the lattice points and only sites with weights greater than 50 are shown. It is apparent that more lattice points exist with higher weights in the Scheidegger model.

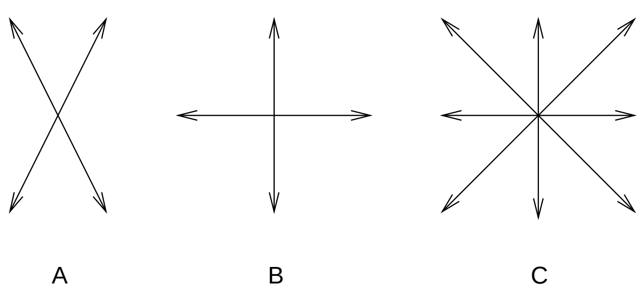

Figure 1. Different lattices that have been used for generating the Scheidegger and Eden networks: the triangular lattice (A), the square lattice (B) and the square lattice extended by the diagonals (C).

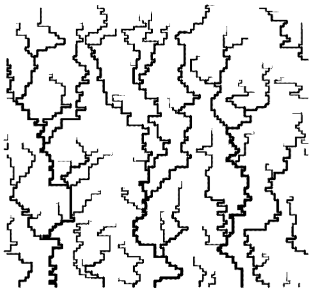

Figure 2. Resulting pattern for the Scheidegger network model on the square lattice B of size 100×100. The thickness of the lines is proportional to the weight of the lattice point. Only sites with weights greater than 50 are shown.

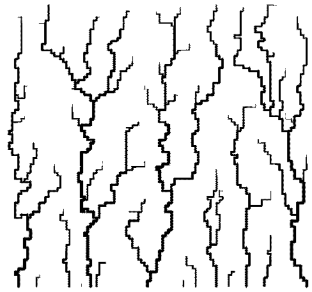

Figure 3. An example of an Eden network on the square lattice B of size 100×100. The thickness of the lines is proportional to the weight of the lattice point. Only sites with weights greater than 50 are shown.

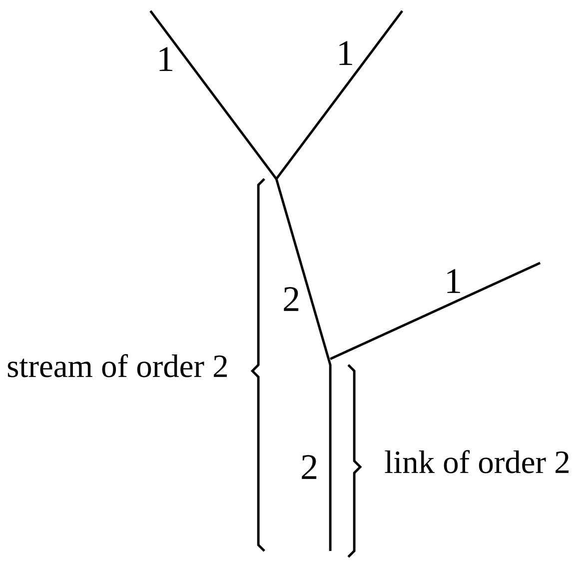

Figure 4. A network of order $\Omega = 2$ with links labelled by their Strahler order.

Since both models have also been proposed in the context of river networks we will adopt that nomenclature. Paths will be called streams, starting points will be called sources and a part of a stream between two junctions or between a source and the next junction will be called a link. In this context we can also identify the weights as drainage areas.

2.2. Horton's laws

The topology of branched structures is often characterized in terms of Horton's law of stream numbers and stream length. For the measurement of the laws one has to order the network. We used the Strahler ordering scheme (Strahler 1964) which is defined as follows (figure 4).

All streams without tributaries are first order. Where two streams of equal order ω join they terminate and a stream of order $\omega + 1$ begins. Where two streams of different order meet the lower-order stream terminates and the higher-order stream continues through the junction. The stream with the highest order determines the order Ω of the network.

Horton's (Horton 1945) law of stream numbers states that N_ω, the number of streams of order ω, decreases geometrically with stream order,

$$\frac{N_\omega}{N_{\omega+1}} = R_B \qquad \text{or} \qquad N_\omega = R_B^{\Omega-\omega}. \tag{1}$$

R_B is called the bifurcation ratio. Horton's law of stream length holds that the mean length L_ω of streams of order ω increases geometrically with stream order,

$$\frac{L_\omega}{L_{\omega-1}} = R_L \qquad \text{or} \qquad L_\omega = L_1 R_L^{\omega-1}. \tag{2}$$

L_1 denotes the mean length of streams of order 1 and R_L is called the length ratio. The Horton constants R_B and R_L determine the topological (similarity) dimension D_t of the network (La Babera and Rosso 1989, Liu 1992):

$$D_t = \frac{\ln R_B}{\ln R_L}. \tag{3}$$

The scale invariance of the self-affine Scheidegger and Eden networks could also be observed by measuring the distribution of the weights of the lattice points. The integrated distribution function $P(A > a)$ will asymptotically decrease according to a power law

$$P(A > a) \sim a^{-\alpha}. \tag{4}$$

For the Scheidegger model on the triangular lattice, the exponent α can be calculated from the critical properties of a one-dimensional random walk and equals $\frac{1}{3}$ (Kondoh *et al* 1987, Takayasu *et al* 1988) (cf section 4).

Under the assumption that all links of the network occupy in the mean the same area one can relate the topological dimension D_t to the exponent α, namely (de Vries *et al* 1994)

$$\alpha = 1 - \frac{1}{D_t} = 1 - \frac{\ln R_L}{\ln R_B}. \tag{5}$$

In the case of the self-similar trees introduced by Mandelbrot (1983) one can obtain this by a direct calculation (de Vries and Eckhardt 1996).

The value of α is also determined by the values of $\nu_\perp$ and $\nu_\parallel$ which describe the scaling of the width W and the length L of the network with the area A,

$$W \sim A^{\nu_\perp} \tag{6}$$
$$L \sim A^{\nu_\parallel}. \tag{7}$$

For a self-affine network $\nu_\perp$ differs from $\nu_\parallel$. Since for compact networks the area A is proportional to WL we have

$$\nu_\perp + \nu_\parallel = 1. \tag{8}$$

Furthermore, in this case we have the relation (Matsushita and Meakin 1988)

$$\alpha = \nu_\perp. \tag{9}$$

2.3. *Multifractal scaling*

Another interesting property to study is the multifractal structure of the drainage area distribution or equally the distribution of the weights $a(i)$ (Nagatani 1993a). Consider a network on a finite lattice of $N \times N$ points. To avoid finite-size effects we will only take into account lattice points with weights less than M where M is of the order of N. Let a_i be the weight of lattice point i. We introduce the partition function $Z_M(q)$ as the sum over all weights $a_i \leqslant M$ raised to the power of q (Nagatani 1993a),

$$Z_M(q) = \sum_{a_i \leqslant M} a_i^q. \tag{10}$$

With $m(a)$, the number of sites with weight a, we rewrite the partition function as

$$Z_M(q) = \sum_{a=1}^{M} m(a) a^q. \tag{11}$$

In analogy to statistical thermodynamics one can define a 'free energy' $F_M(q)$,

$$F_M(q) = -\frac{\ln Z_M(q)}{\ln M}. \tag{12}$$

Phase transitions will be identified as non-analycities in the q-dependence of the free energy $F_M(q)$ (Katzen and Procaccia 1987, Bohr *et al* 1988). In the case of an algebraic decay of $m(a) \sim a^{-\alpha-1}$ the free energy $F_M(q)$ shows a first-order phase transition in the limit of $M \to \infty$ at $q = \alpha$. We will show this with the help of the 'specific heat'

$$C_M(q) = \frac{\partial^2}{\partial q^2} F_M(q) \tag{13}$$

which will develop a singularity at this point. With these assumptions,

$$C_M(q) = -\frac{1}{\ln M} \frac{\partial^2}{\partial q^2} \left(\ln \sum_{a=1}^{M} a^{q-\alpha-1} \right). \tag{14}$$

$C_M(q)$ is a single humped function which converges to a delta function at $q = \alpha$ in the limit $M \to \infty$. Within the numerical simulations $C_M(q)$ is calculated with the help of equations (13) and (10). Figure 6(a) shows some examples of $C_M(q)$ for M for the Scheidegger model on the triangular lattice for $M = 500, 1000, 2000, 4000$. One can already see that the convergence is only logarithmic since the differences between successive minima are constant.

The finite-size scaling behaviour of the specific heat $C_M(q)$ is also accessible. Replacing the sum $\sum_{a=1}^{M} a^{q-\alpha-1}$ by an integral gives

$$C_M(q) = \ln M \left(\frac{M^{q-\alpha}}{(M^{q-\alpha} - 1)^2} - \frac{1}{((q-\alpha)\ln M)^2} \right). \tag{15}$$

With the new variable

$$y = (q - \alpha) \ln M \tag{16}$$

the specific heat becomes

$$C_M(q) = \ln(M) g(y) \tag{17}$$

with the scaling function

$$g(y) = \frac{e^y}{(e^y - 1)^2} - \frac{1}{y^2}. \tag{18}$$

This shows that for different M the functions $C_M(q)$ in (15) can be mapped onto the function $g(y)$ by rescaling the axes. The convergence to the limiting function is only logarithmic in the parameter M characterizing the system size.

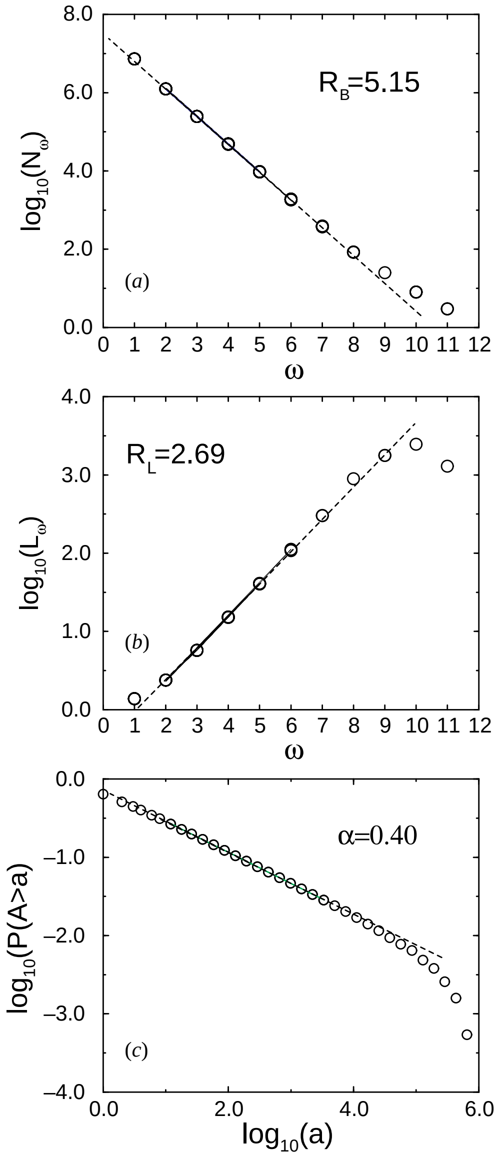

Figure 5. Horton diagrams and area distribution for an Eden network on a 3000×3000 square lattice. The diagrams show the distribution of (*a*) stream numbers, (*b*) stream length and (*c*) integrated area distribution. The regions in which the scaling exponents were fitted are indicated by full lines while the dotted lines indicate the extension over the full range.

3. Numerical results

For the measurement of the Horton constants R_B and R_L we generated for every lattice type 10 networks of size 3000×3000. Figures 5(*a*) and (*b*) show the Horton diagrams for an Eden network on a 3000×3000 square lattice. On this lattice there are many streams, the largest of which has an order of about 11 (this is the typical value; in a few realizations a maximal order 12 was observed). To increase the maximal order Ω by 1 requires roughly doubling the lattice size. In both networks streams of order one do not fit the Horton laws very well; they are too numerous and too long. This is also observed in DLA clusters and river networks (Yekutieli *et al* 1994, Abrahams 1984). To avoid influences from large and small streams we measured the Horton constants R_B and R_L between order 2 and 5 only.

Table 1. Horton constants and the exponents α for the Scheidegger and Eden networks. The measured values in columns 3–5 represent averages over 10 networks of size 3000×3000. α_{exp} was determined by a fit according to equation (4).

Model	Lattice	Measured α_{exp}	R_L	R_B	α from equation (5)
Scheidegger	A	0.336 ± 0.004	2.94 ± 0.02	5.12 ± 0.02	0.339 ± 0.007
Scheidegger	B	0.334 ± 0.003	2.95 ± 0.01	5.11 ± 0.02	0.336 ± 0.005
Scheidegger	C	0.335 ± 0.004	2.95 ± 0.02	5.15 ± 0.03	0.34 ± 0.01
Eden	A	0.398 ± 0.004	2.70 ± 0.02	5.15 ± 0.02	0.394 ± 0.008
Eden	B	0.396 ± 0.004	2.69 ± 0.02	5.13 ± 0.02	0.395 ± 0.009
Eden	C	0.400 ± 0.004	2.69 ± 0.02	5.14 ± 0.02	0.395 ± 0.009

For every network we also measured the integrated distribution function $P(A > a)$ (see figure 5(c) for an Eden network). It decays algebraically with an exponent α over several decades.

The results of our measurements are collected in table 1. Within the standard error the observable are independent of the underlying lattice. For the bifurcation ratio R_B we measured a value of about 5.15 for both network types, which has also been observed for DLA clusters (Yekutieli *et al* 1994, Hinrichsen *et al* 1989). The value of $R_L \approx 2.69$ for the Eden networks is lower than the one for the Scheidegger networks ($R_L \approx 2.95$) since in the former there are more possibilities for the branches to join (compare also figures 2 and 3). The measured values for α according to equation (4) are about 0.33 for the Scheidegger networks and about 0.40 for the Eden networks; they are in good agreement with (5). As far as the scaling exponent $\alpha \approx 0.40$ is concerned the Eden model falls into the universality class of several other network models (Meakin 1987, Meakin *et al* 1991, Leheny 1995, Manna and Subramanian 1996).

In order to measure α with the help of the multifractal distribution of the weights we generated five networks of size 4000×4000 with the Scheidegger model on the square lattice and take into account all lattice points with weights less than or equal to $M = 500, 1000, 2000, 4000$. The specific heat $C_M(q)$ calculated with the help of equation (13) and (10) is shown in figure 6(a). The data points were calculated in steps of $\Delta q = 0.01$. One can already see that the absolute value of the minimum of $C_M(q)$ increases with $\ln M$ since the differences of the minima between successive curves are constant. A measurement of the mean position of the minimum value of $C_M(q)$ leads to estimates of $\alpha = 0.343$, 0.338, 0.335, 0.333 with an uncertainty of 1 in the last digit for $M = 500, 1000, 2000, 4000$ respectively. The uncertainty of the whole function is less than 1%. Figure 6(b) shows the rescaled spectra out of figure 6(a) together with the scaling function $g(y)$. The rescaling was done with a value of $\alpha = \frac{1}{3}$ for all spectra. The rescaled spectra coincide very well which proves the logarithmic convergence towards the delta function. The difference to the scaling function $g(y)$ is mainly caused by the fact that we used an integral approximation for its calculation. The rescaling of a discrete summation is indistinguishable from the results for the networks.

4. Analytical results for the Scheidegger model

In this section we want to calculate the exact values of the area distribution $p(a)$ for the Scheidegger model on the triangular lattice for small areas a. As already mentioned, the asymptotic value of $\alpha = \frac{1}{3}$ can be calculated from the critical properties of a simple random

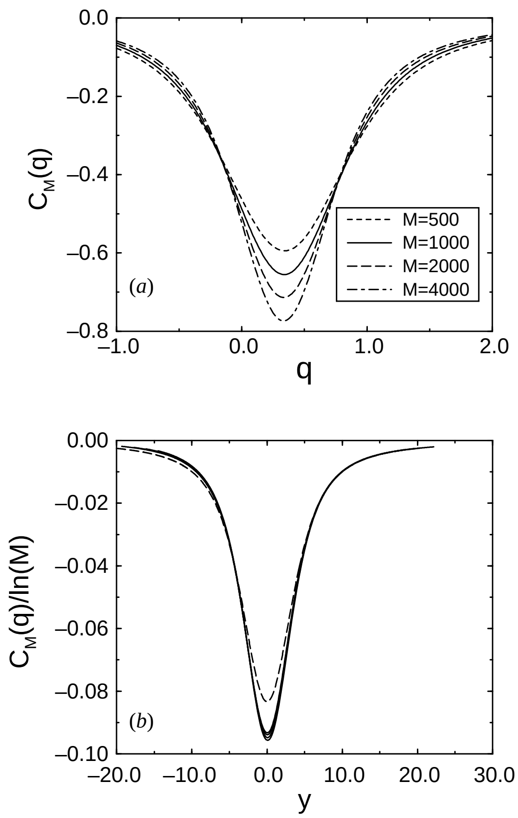

Figure 6. Specific heats $C_M(q)$ for the Scheidegger models on a triangular lattice for different sizes M. (a) Unscaled data and (b) rescaled using (15) and (16) with $\alpha = \frac{1}{3}$.

walk (Kondoh *et al* 1987, Takayasu *et al* 1988). Since the critical properties of directed random walks on lattice B and C are the same (Redner and Majid 1983) the asymptotic value of α should also equal $\frac{1}{3}$. This is supported by our numerical results.

For the triangular lattice the boundary of a network is equivalent to the traces of a pair of random walkers s and r who start at a distance 1 apart and join together for the first time at the top of the network (figure 7). The distance $x := r - s$ between the two random walkers obeys the equations for a random walk that starts at time $t = 0$ at position 1 and changes in the next time step to 1 ± 1 with probability $\frac{1}{4}$ and keeps its position with probability $\frac{1}{2}$ (figure 8). The walk stops when $x = 0$ is reached (the two walkers meet again).

Let $\boldsymbol{w}(t) = (w_1, \ldots, w_i, \ldots)$ be the vector of probabilities for the random walker to reach site i at time t. Then the probabilities in the next time step are given by $\boldsymbol{w}(t+1) = \mathsf{T}\boldsymbol{w}(t)$ with the transfer matrix

$$\mathsf{T} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{w}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{19}$$

In order to keep track of the area enclosed by the random walker we introduce a book keeping variable z, so that its power counts the enclosed are. The entries of the vector

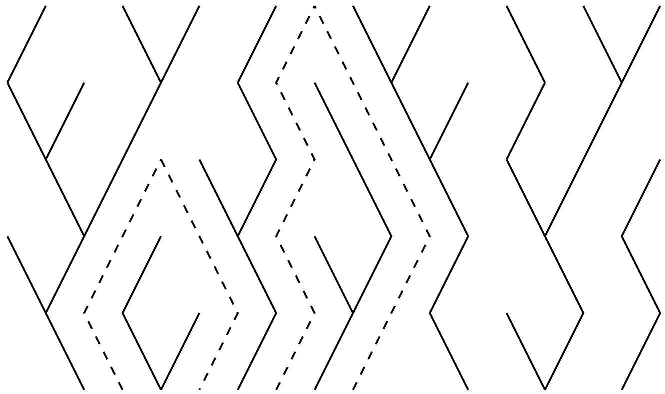

Figure 7. Area relations in the Scheidegger model. The total area drained by a network is bounded by the dotted lines. The enclosed area can be estimated from random walks that start one lattice site apart and meet again for the first time at the top of the network.

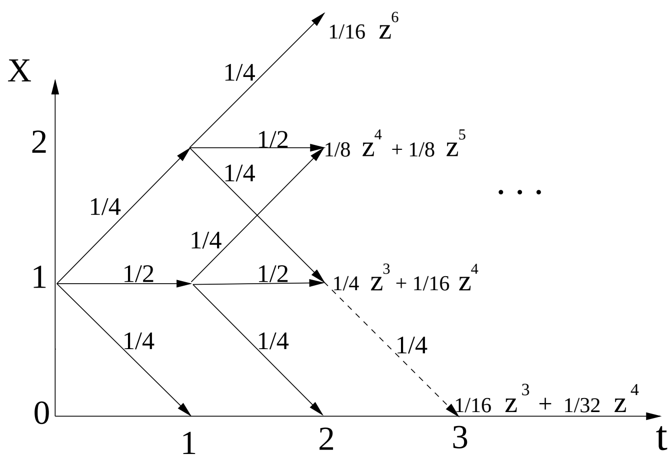

Figure 8. Random walks, enclosed areas and generating functions for the Scheidegger model. See text for a further explanation.

$\boldsymbol{w}'(t)$ will now be polynomials in z and the transfer matrix is modified to

$$\mathsf{T}' = \begin{pmatrix} \frac{1}{2}z & \frac{1}{4}z & 0 & 0 & 0 & \cdots \\ \frac{1}{4}z^2 & \frac{1}{2}z^2 & \frac{1}{4}z^2 & 0 & 0 & \cdots \\ 0 & \frac{1}{4}z^3 & \frac{1}{2}z^3 & \frac{1}{4}z^3 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}. \qquad (20)$$

Consider, as an example, the first few steps indicated in figure 8. The random walker starts at a distance 1, carrying a single site as weight. Thus $\boldsymbol{w}'(0) = (z, 0, \ldots)$. After the first iteration $\boldsymbol{w}'(1) = (\frac{1}{2}z^2, \frac{1}{4}z^3, 0, \ldots)$. This accounts for the two possibilities: 'staying at a site and adding area 1' and 'moving one up and adding area 2'. After the second iteration, $\boldsymbol{w}'(2) = (\frac{1}{4}z^3 + \frac{1}{16}z^4, \frac{1}{8}z^4 + \frac{1}{8}z^5, \frac{1}{16}z^6, \ldots)$. The entries at $i = 1$ now contain the contributions from the paths that stayed at $i = 1$ and accumulated an area of 3 and the one that stepped out and back, accumulating an area 4. The probabilities we are interested

Table 2. Results of the calculation of the exact probabilities $p(a)$ together with the values of the locally defined $\alpha(a)$.

a	$p(a)$	$\alpha(a)$
1	$\frac{1}{4}$	0.263 03
2	$\frac{1}{8}$	0.259 85
3	$\frac{1}{16}$	0.302 46
4	$\frac{3}{64}$	0.280 18
5	$\frac{1}{32}$	0.318 71
6	$\frac{7}{256}$	0.297 82
7	$\frac{21}{1024}$	0.316 54
8	$\frac{37}{2048}$	0.312 80
9	$\frac{31}{2048}$	0.316 92
10	$\frac{217}{16\,384}$	
19	$\frac{376\,401}{67\,108\,864}$	0.325 15
20	$\frac{704\,727}{134\,217\,728}$	
29	$\frac{219\,605\,077}{68\,719\,476\,736}$	0.327 71
30	$\frac{3359\,578\,083}{1099\,511\,627\,776}$	

in are contained in the first element $i = 1$, the probabilities for a certain area being the coefficient of the corresponding power in z.

To get the desired probabilities $p(a)$ we have to sum up all coefficients of the polynomials in the first element of the vectors $\boldsymbol{w}'(k)$, $k = 0, \ldots, a$, with a degree equal to a and multiply the final sum with $\frac{1}{4}$ realizing the die out of the networks. A transfer matrix of size $N \times N$ yields the probabilities $p(a)$ up to $a = N^2$.

We calculated the $p(a)$ up to $a = 36$ with the help of MAPLE and studied the convergence of the locally defined $\alpha(a)$

$$\alpha(a) := \frac{\log p(a) - \log p(a+1)}{\log(a+1) - \log(a)} - 1 \tag{21}$$

towards the exact value of $\frac{1}{3}$ (table 2). For $a > 23$ the $\alpha(a)$ converge monotonically towards the limiting value of $\frac{1}{3}$. The value of $\alpha(35)$ is already less than 2% away from this value. As shown in figure 9(a) the differences between the $\alpha(a)$ and the limit value of $\frac{1}{3}$ decay like $\frac{1}{3} - \alpha(a) \sim a^{-1}$. Assuming this convergence we can estimate the limit value α_∞ from pairs $\alpha(a), \alpha(a+1)$ (figure 9(b)). This also converges to $\frac{1}{3}$, supporting the scaling hypothesis.

5. Final remarks

The Scheidegger and Eden model belong to different universality classes as far as the scaling exponent α is concerned. But this difference seems to be due to a difference in the Horton constant R_L only.

For the multifractal spectrum of the area distribution we could show the existence of a phase transition point in the limit of $M \to \infty$ where M stands for the system size. The transition point is defined by the exponent α in the integrated area distribution $P(Q > q) \sim q^{-\alpha}$. The approach to the phase transition for finite M is very slow and controlled by $\ln M$.

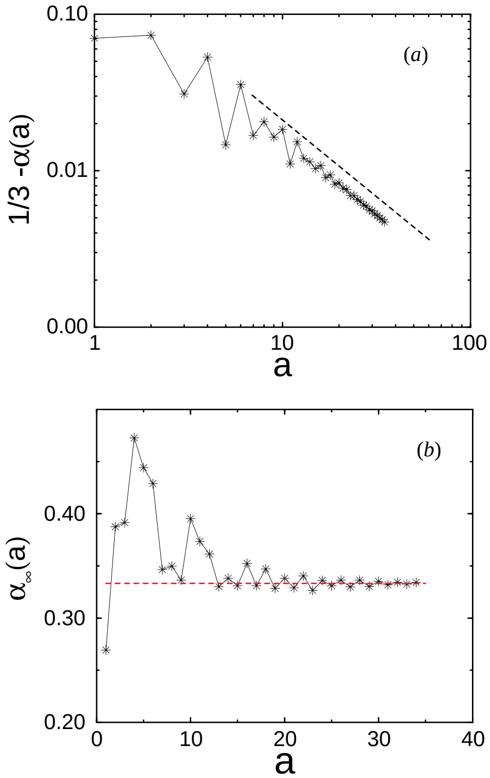

Figure 9. Finite-size scaling for the exponent α in Scheidegger networks. (a) The differences $\frac{1}{3} - \alpha(a)$ scale like a^{-1}. (b) Assuming the above scaling, the locally estimated asymptotic values α_∞ also approach $\frac{1}{3}$.

In a stimulating recent contribution, Manna and Dhar (1996) related exponents in the Eden model to exponents in the Kardar, Parisi, Zhang equation for the growth of interfaces (Kardar *et al* 1986, Halphin-Healy and Zhang 1995). In particular, the length L of the network and the corresponding width W should scale like $W \sim L^{2/3}$. From the definition of $\nu_\perp$ and $\nu_\|$ in (6) and (7) it follows that $W \sim L^{\nu_\perp/\nu_\|}$. Since the networks are compact, $\nu_\perp + \nu_\| = 1$ and thus $\nu_\perp = \frac{2}{5}$. Using (9) and (5) we find $\alpha = \frac{2}{5}$ and $D_t = \frac{5}{3}$. Perhaps the analysis presented here can shed light on the universality classes and finite-size scaling for interface growths as well.

Acknowledgment

We thank Pal Rujan for fruitful discussions on the transfer matrix method.

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